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BOUNDS ON PLASTIC DEFORMATIONS OF TRUSSES

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Abstract—A technique to compute bounds on plastic deformations of truss structures is presented. Each element of the structure is constituted by an elastic–perfectly plastic material. The structure is subjected to a combination of loadings (i.e. a steady mechanical load and a cyclic mechanical and/or kinematical load) resulting slightly above the shakedown limit. The proposed technique utilizes a bounding principle by means of which it is possible to calculate a bound on a special proportionality factor. This characterizes the relation between the kinematical part of the solution to the Kuhn–Tucker equations relative to the shakedown cyclic load factor problem and the gradient (with respect to the cyclic load multiplier) of the kinematical part of the elastic–plastic steady-state response of the structure to loads at the shakedown limit. Bounds on any measure of real plastic deformation, in the structure subjected to the combination of loadings slightly above the shakedown limit, can be then easily computed by means of the knowledge of the bound on the above-mentioned proportionality factor as well as of the kinematical part of the solution to the shakedown cyclic load factor problem.

1. INTRODUCTION

It is known that structures are often subjected to a combination of steady and cyclic loads. For such a loading case some approximate step-by-step analysis processes (Polizzotto, 1989), and several bounding techniques (Capurso *et al.*, 1979; König, 1979; Polizzotto, 1982; Giambanco *et al.*, 1990a), devoted to the search of an approximate response of the structure, have already been presented. Panzeca and Polizzotto (1988) showed that the kinematical part of the solution to the shakedown cyclic load factor problem is proportional to the gradient, with respect to the cyclic load multiplier, of the elastic–plastic steady-state response of the structure to loads at the shakedown limit. On the grounds of such a relationship, in the framework of elastic–perfectly plastic bodies, Giambanco *et al.* (1990b, 1992) developed a bounding technique for plastic deformations whose main step is the computation of a bound on the proportionality factor between the two above solutions at the shakedown limit. The knowledge of such a bound allows us to compute bounds on any measure of real plastic deformation for loads slightly above the shakedown limit, by making use of the solution to the shakedown cyclic load factor problem.

Such formulations, devoted to continuous bodies, have a great theoretical interest but present several difficulties in order to be applied. For computational purposes the derivation of an analogous formulation devoted to discrete or discretized structures, and therefore the derivation of an appropriate form for the bounding principle, is desirable.

In order to achieve this aim the previously described technique is specialized to elastic–perfectly plastic truss structures; the obtained results can then be readily generalized to discrete structures, in general, as well as to finite element models of more complex structures and bodies.

The equations governing the steady-state elastic–plastic response and the gradient, with respect to the cyclic load multiplier, of such a response at the shakedown limit are derived. Also, the shakedown load factor problem and the related Kuhn–Tucker equations are specialized in the case of truss structures. Finally, making use of a suitably specialized bounding principle and on the grounds of the proportionality between the solutions to the two relevant problems one is able to compute bounds on any chosen measure of real plastic deformation for loads slightly above the shakedown limit.

2. THE ELASTIC–PLASTIC MODEL

Let us consider a truss, subjected to the actions of a nodal force vector $\mathbf{f}$ and of an element imposed strain vector $\mathbf{q}_\theta$, both varying in time according to a specified load history (quasi-static load). The material being elastic–plastic, the truss elements generally suffer elastic and plastic deformations, so the total strain $\mathbf{q}$ can be decomposed into an elastic, an imposed, and a plastic part, namely

$$\mathbf{q} = \mathbf{q}_e + \mathbf{q}_\theta + \mathbf{q}_p. \tag{1}$$

On denoting by $\mathbf{Q}$ the element stress vector and by $\boldsymbol{\eta}$ the structure node displacement vector, in the hypothesis of small displacements, the equations governing the truss response are as follows:

$$\mathbf{q} = \mathbf{C}\boldsymbol{\eta} \quad \text{(compatibility)} \tag{2a}$$

$$\mathbf{Q} = \mathbf{F}^{-1}\mathbf{q}_e \quad \text{(elasticity law)} \tag{2b}$$

$$\mathbf{C}^T\mathbf{Q} = \mathbf{f} \quad \text{(equilibrium)}, \tag{2c}$$

where $\mathbf{C}$ is the compatibility matrix, $\mathbf{F}^{-1}$ is the internal stiffness matrix and the superscript T denotes the transpose of the relevant quantity. By eqns (1) and (2a–c), one obtains

$$\mathbf{S}\boldsymbol{\eta} = \hat{\mathbf{f}}, \tag{2d}$$

where

$$\mathbf{S} = \mathbf{C}^T\mathbf{F}^{-1}\mathbf{C} \quad \text{and} \quad \hat{\mathbf{f}} = \mathbf{f} + \mathbf{C}^T\mathbf{F}^{-1}\mathbf{q}_\theta + \mathbf{C}^T\mathbf{F}^{-1}\mathbf{q}_p \tag{2e}$$

are the external stiffness matrix and the equivalent node force vector, respectively. Given the vector $\mathbf{q}_p$, eqn (2d) provides the structure node displacements. Otherwise, in addition to eqns (2), the equations governing the plastic behaviour of the truss must be taken into account; they are

$$\boldsymbol{\varphi} = \mathbf{N}^T\mathbf{Q} - \mathbf{Q}_y \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \boldsymbol{\varphi}^T\dot{\boldsymbol{\lambda}} = 0 \tag{3a}$$

$$\dot{\mathbf{q}}_p = \mathbf{N}\dot{\boldsymbol{\lambda}}, \tag{3b}$$

where $\mathbf{Q}_y$ is the element yield stress vector, $\mathbf{N}^T$ is the matrix of unit external normals to the element elastic domains, $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{Q})$ is the plastic potential vector, $\dot{\boldsymbol{\lambda}}$ is the plastic multiplier vector, while a superimposed dot indicates the relevant quantity time rate.

3. THE STEADY-STATE RESPONSE TO CYCLIC LOADS

Let us suppose now that the structure be subjected to the contemporaneous action of a steady mechanical load and a cyclic mechanical and/or kinematical load. Let us denote by $\mathbf{f}_0$ the time independent part of the node force vector, by $\mathbf{f}_c = \mathbf{f}_c(t)$ the cyclic part of the node force vector, by $\mathbf{q}_\theta = \mathbf{q}_\theta(t)$ the cyclic element imposed strain vector, by $\hat{\mathbf{f}}_c = \mathbf{f}_c + \mathbf{C}^T\mathbf{F}^{-1}\mathbf{q}_\theta$ the cyclic equivalent node force vector, by $t \geqslant 0$ the time variable and by Δt the time period [$\mathbf{f}_c(t)$ and $\mathbf{q}_\theta(t)$ vary with the same time period Δt]. Let $\boldsymbol{\eta}_0, \mathbf{Q}_0$ and $\boldsymbol{\eta}_c, \mathbf{Q}_c$ denote the elastic response to the reference loads $\mathbf{f}_0$ and $\hat{\mathbf{f}}_c$, respectively.

Let ξ_0, $0 \leqslant \xi_0 \leqslant \xi_L$, denote a parameter multiplier of $\mathbf{f}_0$, such that for all values of $\xi_0 < \xi_L$ the load $\xi_0\mathbf{f}_0$ is below the plastic collapse limit value, while $\xi_L\mathbf{f}_0 = \mathbf{f}_L$ is the ultimate limit load. On denoting by ξ a multiplier of $\hat{\mathbf{f}}_c$, for any selected value $\bar{\xi}_0$ of the parameter ξ_0, a family of cyclic loads $\hat{\mathbf{f}}_\xi = \bar{\xi}_0\mathbf{f}_0 + \xi\hat{\mathbf{f}}_c$ could be generated. The maximum value $X_E = X_E(\xi_0)$ of the multiplier ξ, for which for any $\xi < X_E$ the truss behaves elastically, is the elastic limit multiplier of the given structure. The maximum value $X = X(\xi_0)$ of the multiplier ξ, for which for any $\xi < X$ the truss shows an elastic behaviour during the steady-state phase, eventually after an initial transient phase in which some plastic deformations can be produced, is the shakedown limit load factor of the given structure. The curves $X_E(\xi_0)$ and $X(\xi_0)$, together with the curve that separates the alternating plasticity zone from the incremental collapse zone, constitute, in the (ξ_0, ξ)-plane, the so-called Bree diagram (Polizzotto, 1989).

For $\xi > X$, shakedown does not occur and the steady-state response of the structure is elastic–plastic. It is known from the cyclic plasticity theory (Martin, 1975) that an elastic–plastic structure subjected to cyclic loads shows a non-periodic response during the initial transient phase, while after, lasting a certain number of periods, the structure response becomes steady-state and the generalized stress vector $\mathbf{Q}$, as well as the generalized plastic strain rate vector $\dot{\mathbf{q}}_p$, becomes cyclic, with the same period Δt of the loads. Then, for any arbitrary instant t_0 subsequent to the transient interval, one has

$$\mathbf{Q}(t_0) = \mathbf{Q}(t_0+\Delta t), \qquad \dot{\mathbf{q}}_p(t_0) = \dot{\mathbf{q}}_p(t_0+\Delta t). \tag{4}$$

The generalized stress vector $\mathbf{Q}$ appearing in eqns (4) can be understood as the sum of the elastic response to the steady load, the elastic response to the cyclic load, and a residual (self-equilibrated) generalized stress vector $\mathbf{Q}_r$, namely

$$\mathbf{Q}(t) = \bar{\xi}_0\mathbf{Q}_0+\bar{\xi}\mathbf{Q}_c(t)+\mathbf{Q}_r(t) = \bar{\xi}_0\mathbf{F}^{-1}\mathbf{C}\boldsymbol{\eta}_0+\bar{\xi}(\mathbf{F}^{-1}\mathbf{C}\boldsymbol{\eta}_c-\mathbf{F}^{-1}\mathbf{q}_\theta)+\mathbf{Q}_r \tag{5}$$

being $\bar{\xi}_0 < \xi_L$ and $\bar{\xi} > X$ two selected load multipliers, determining the load $\hat{\mathbf{f}}_\xi = \bar{\xi}_0\mathbf{f}_0+\bar{\xi}\hat{\mathbf{f}}_c$.

On denoting by t an instant subsequent to t_0 and by $\tau = t-t_0$ an auxiliary time variable within the range $(0, \Delta t)$, the relationships governing the steady-state phase can be written in the following form

$$\boldsymbol{\varphi} = \mathbf{N}^T\mathbf{Q}-\mathbf{Q}_y \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \boldsymbol{\varphi}^T\dot{\boldsymbol{\lambda}} = 0, \quad \forall\,\tau\in(0,\Delta t) \tag{6a}$$

$$\mathbf{S}\boldsymbol{\eta}_0-\mathbf{f}_0 = \mathbf{0} \tag{6b}$$

$$\mathbf{S}\boldsymbol{\eta}_c-\hat{\mathbf{f}}_c = \mathbf{0}, \quad \forall\,\tau\in(0,\Delta t) \tag{6c}$$

$$\mathbf{C}^T\mathbf{Q}_r = \mathbf{0}, \quad \forall\,\tau\in(0,\Delta t) \tag{6d}$$

$$\dot{\mathbf{q}}_p = \mathbf{N}\dot{\boldsymbol{\lambda}}, \quad \forall\,\tau\in(0,\Delta t) \tag{6e}$$

$$\dot{\mathbf{q}}_r = \mathbf{F}\dot{\mathbf{Q}}_r+\dot{\mathbf{q}}_p, \quad \forall\,\tau\in(0,\Delta t) \tag{6f}$$

$$\dot{\mathbf{q}}_r = \mathbf{C}\dot{\boldsymbol{\eta}}_r, \quad \forall\,\tau\in(0,\Delta t) \tag{6g}$$

$$\int_0^{\Delta t}\dot{\mathbf{Q}}_r\,\mathrm{d}\tau = \mathbf{0}, \tag{6h}$$

where $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{Q}(t))$ is the yield function vector, $\dot{\boldsymbol{\lambda}} = \dot{\boldsymbol{\lambda}}(t)$ is the plastic multiplier vector, $\dot{\mathbf{Q}}_r = \dot{\mathbf{Q}}_r(t)$, $\dot{\boldsymbol{\eta}}_r = \dot{\boldsymbol{\eta}}_r(t)$ and $\dot{\mathbf{q}}_r = \dot{\mathbf{q}}_r(t)$ denote the residual stress, displacement and strain vector, respectively. Equations (6f–h) imply that the plastic accumulation in the cycle

$$\Delta\mathbf{q}_p = \int_0^{\Delta t}\dot{\mathbf{q}}_p\,\mathrm{d}\tau \tag{7}$$

is a vector independent of t_0 and compatible with the displacement increments in the cycle

$$\Delta\boldsymbol{\eta}_r = \int_0^{\Delta t}\dot{\boldsymbol{\eta}}_r\,\mathrm{d}\tau \tag{8}$$

i.e.

$$\Delta\mathbf{q}_p = \mathbf{C}\Delta\boldsymbol{\eta}_r. \tag{9}$$

The solution to eqns (6a–h) or (6a–g) and (7)–(9) is unique and the plastic dissipation rate

$$\dot{D} = \mathbf{Q}^T \dot{\mathbf{q}}_p \tag{10}$$

turns out to be uniquely determined $\forall\, \tau \in (0, \Delta t)$.

If some component of the plastic accumulation vector $\Delta \mathbf{q}_p$ different to zero exists, ratchetting occurs; otherwise, if $\Delta \mathbf{q}_p = \mathbf{0}$, low cycle fatigue of the material is likely to be induced.

4. THE STEADY-STATE RESPONSE TO CYCLIC LOADS AT THE SHAKEDOWN LIMIT

When (still provided that $\xi > X$) the multiplier $\xi \to X^+$, the limit response is characterized by a plastic strain rate vector and a residual stress rate vector vanishing identically, namely

$$\lim_{\xi \to X^+} \dot{\mathbf{q}}_p = \mathbf{0}, \qquad \lim_{\xi \to X^+} \dot{\mathbf{Q}}_r = \mathbf{0}, \qquad \forall\, \tau \in (0, \Delta t), \tag{11}$$

so that the residual stresses set up a time-independent self-stress vector, i.e.

$$\lim_{\xi \to X^+} \mathbf{Q}_r(t) = \mathbf{Q}_\rho^+. \tag{12}$$

Equations (6d–h) and (7)–(9) become identities. However, by introducing the new quantities

$$\dot{\boldsymbol{\lambda}}^+ = \lim_{\xi \to X^+} \frac{\dot{\boldsymbol{\lambda}}}{\xi - X} \tag{13a}$$

$$\dot{\mathbf{q}}_p^+ = \lim_{\xi \to X^+} \frac{\dot{\mathbf{q}}_p}{\xi - X}, \qquad \dot{\mathbf{q}}_r^+ = \lim_{\xi \to X^+} \frac{\dot{\mathbf{q}}_r}{\xi - X} \tag{13b}$$

$$\dot{\boldsymbol{\eta}}_r^+ = \lim_{\xi \to X^+} \frac{\dot{\boldsymbol{\eta}}_r}{\xi - X}, \qquad \dot{\mathbf{Q}}_r^+ = \lim_{\xi \to X^+} \frac{\dot{\mathbf{Q}}_r}{\xi - X} \tag{13c}$$

$$\dot{D}^+ = \lim_{\xi \to X^+} \frac{\dot{D}}{\xi - X}, \tag{13d}$$

and setting

$$\mathbf{Q}^+ = \xi_0 \mathbf{Q}_0 + X \mathbf{Q}_c + \mathbf{Q}_\rho^+ = \xi_0 \mathbf{F}^{-1} \mathbf{C} \boldsymbol{\eta}_0 + X(\mathbf{F}^{-1} \mathbf{C} \boldsymbol{\eta}_c - \mathbf{F}^{-1} \mathbf{q}_\theta) + \mathbf{Q}_\rho^+, \tag{13e}$$

eqns (6a–h) and (7)–(9) take the form

$$\boldsymbol{\varphi}^+ = \mathbf{N}^T \mathbf{Q}^+ - \mathbf{Q}_y \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}^+ \geqslant \mathbf{0}, \quad (\boldsymbol{\varphi}^+)^T \dot{\boldsymbol{\lambda}}^+ = 0, \quad \forall\, \tau \in (0, \Delta t) \tag{14a}$$

$$\mathbf{S} \boldsymbol{\eta}_0 - \mathbf{f}_0 = \mathbf{0} \tag{14b}$$

$$\mathbf{S} \boldsymbol{\eta}_c - \hat{\mathbf{f}}_c = \mathbf{0}, \quad \forall\, \tau \in (0, \Delta t) \tag{14c}$$

$$\mathbf{C}^T \mathbf{Q}_\rho^+ = \mathbf{0} \tag{14d}$$

$$\dot{\mathbf{q}}_p^+ = \mathbf{N} \dot{\boldsymbol{\lambda}}^+, \quad \forall\, \tau \in (0, \Delta t) \tag{14e}$$

$$\dot{\mathbf{q}}_r^+ = \mathbf{F} \dot{\mathbf{Q}}_r^+ + \dot{\mathbf{q}}_p^+, \quad \forall\, \tau \in (0, \Delta t) \tag{14f}$$

$$\dot{\mathbf{q}}_r^+ = \mathbf{C} \dot{\boldsymbol{\eta}}_r^+, \quad \forall\, \tau \in (0, \Delta t) \tag{14g}$$

$$\int_0^{\Delta t} \dot{\mathbf{Q}}_r^+ \, \mathrm{d}\tau = 0 \tag{14h}$$

$$\Delta \mathbf{q}_p^+ = \int_0^{\Delta t} \dot{\mathbf{q}}_p^+ \,\mathrm{d}\tau \tag{15}$$

$$\Delta \boldsymbol{\eta}_r^+ = \int_0^{\Delta t} \dot{\boldsymbol{\eta}}_r^+ \,\mathrm{d}\tau \tag{16}$$

$$\Delta \mathbf{q}_p^+ = \mathbf{C}\Delta \boldsymbol{\eta}_r^+. \tag{17}$$

Equations (14) and (15)–(17) describe a deformation process for which the gradient of external work in the cycle, for loads at the shakedown limit, is given by

$$E^+ = E_0^+ + E_c^+ = \xi_0 \int_0^{\Delta t} \mathbf{f}_0^T \dot{\boldsymbol{\eta}}_r^+ \,\mathrm{d}\tau + X \int_0^{\Delta t} \hat{\mathbf{f}}_c^T \dot{\boldsymbol{\eta}}_r^+ \,\mathrm{d}\tau = \xi_0 \mathbf{f}_0^T \Delta \boldsymbol{\eta}_r^+ + X \int_0^{\Delta t} \hat{\mathbf{f}}_c^T \dot{\boldsymbol{\eta}}_r^+ \,\mathrm{d}\tau. \tag{18}$$

Because, by virtue of eqns (14f, g), (15) and (17), and by virtue of the virtual work principle

$$\int_0^{\Delta t} \mathbf{f}_0^T \dot{\boldsymbol{\eta}}_r^+ \,\mathrm{d}\tau = \int_0^{\Delta t} \mathbf{Q}_0^T \dot{\mathbf{q}}_r^+ \,\mathrm{d}\tau = \int_0^{\Delta t} \mathbf{Q}_0^T \mathbf{F} \dot{\mathbf{Q}}_r^+ \,\mathrm{d}\tau + \int_0^{\Delta t} \mathbf{Q}_0^T \dot{\mathbf{q}}_p^+ \,\mathrm{d}\tau = \int_0^{\Delta t} \mathbf{Q}_0^T \dot{\mathbf{q}}_p^+ \,\mathrm{d}\tau = \mathbf{Q}_0^T \Delta \mathbf{q}_p^+ \tag{19}$$

$$\int_0^{\Delta t} \hat{\mathbf{f}}_c^T \dot{\boldsymbol{\eta}}_r^+ \,\mathrm{d}\tau = \int_0^{\Delta t} \mathbf{Q}_c^T \dot{\mathbf{q}}_r^+ \,\mathrm{d}\tau = \int_0^{\Delta t} \mathbf{Q}_c^T \mathbf{F} \dot{\mathbf{Q}}_r^+ \,\mathrm{d}\tau + \int_0^{\Delta t} \mathbf{Q}_c^T \dot{\mathbf{q}}_p^+ \,\mathrm{d}\tau = \int_0^{\Delta t} \mathbf{Q}_c^T \dot{\mathbf{q}}_p^+ \,\mathrm{d}\tau \tag{20}$$

$$\int_0^{\Delta t} (\mathbf{Q}_p^+)^T \dot{\mathbf{q}}_p^+ \,\mathrm{d}\tau = (\mathbf{Q}_p^+)^T \Delta \mathbf{q}_p^+ = 0 \tag{21}$$

eqn (18) can be written in the following form

$$\begin{aligned} E^+ = E_0^+ + E_c^+ &= \xi_0 \mathbf{Q}_0^T \Delta \mathbf{q}_p^+ + X \int_0^{\Delta t} \mathbf{Q}_c^T \dot{\mathbf{q}}_p^+ \,\mathrm{d}\tau \\ &= \int_0^{\Delta t} (\xi_0 \mathbf{Q}_0 + X\mathbf{Q}_c + \mathbf{Q}_p^+)^T \dot{\mathbf{q}}_p^+ \,\mathrm{d}\tau = \int_0^{\Delta t} \dot{D}^+(\dot{\mathbf{q}}_p^+)\,\mathrm{d}\tau = D^+ > 0, \end{aligned} \tag{22}$$

where $\dot{D}^+ = \dot{D}^+(\dot{\mathbf{q}}_p^+)$ is the gradient, with respect to the cyclic load multiplier, of the plastic dissipation rate. From the latter it results that the gradient with respect to the cyclic load multiplier of the external work in the cycle, for loads at the shakedown limit E^+, is equal to the gradient of the plastic dissipation in the cycle, D^+.

The constants E_0^+ and E_c^+ can be approximately calculated by the formulae

$$E_0^+ \cong \frac{\xi_0}{\xi - X} \mathbf{Q}_0^T \Delta \mathbf{q}_p, \qquad E_c^+ \cong \frac{X}{\xi - X} \int_0^{\Delta t} \mathbf{Q}_c^T \dot{\mathbf{q}}_p \,\mathrm{d}\tau, \tag{23}$$

where $\dot{\mathbf{q}}_p$ and $\Delta \mathbf{q}_p$ are the real plastic deformations produced by the load $\hat{\mathbf{f}}_\xi = \xi_0 \mathbf{f}_0 + \xi \hat{\mathbf{f}}_c$, being ξ slightly above the shakedown limit.

The constants E_0^+ and E_c^+, from a qualitative point of view, characterize the steady-state plastic behaviour of the truss in presence of cyclic loads slightly above the shakedown limit (Polizzotto, 1989): if $E_0^+ = 0$ and $E_c^+ \neq 0$ plastic shakedown occurs; if $E_0^+ \neq 0$ and $E_c^+ \neq 0$ ratchetting occurs.

5. THE SHAKEDOWN LOAD FACTOR FOR CYCLIC LOADS

The necessary and sufficient conditions under which shakedown occurs are provided by the so-called statical theorem (or Bleich–Melan's theorem). In the relevant case, it states that shakedown occurs if a time-independent self-equilibrated generalized stress vector $\mathbf{Q}_\rho^s$ exists, such that the total generalized stresses, obtained by summing up the self-stress vector and the generalized stress vectors produced by the steady and the cyclic load, do not violate in any instant the plasticity conditions.

Consequently, the shakedown load factor problem identifies itself with the following search problem

$$X = \max_{(\xi, \xi_0, \boldsymbol{\eta}_0, \boldsymbol{\eta}_c, \mathbf{Q}_\rho^s)} \xi \tag{24a}$$

subject to

$$\boldsymbol{\varphi}^s = \mathbf{N}^T\mathbf{Q}^s - \mathbf{Q}_y = \mathbf{N}^T(\xi_0\mathbf{Q}_0 + \xi\mathbf{Q}_c + \mathbf{Q}_\rho^s) - \mathbf{Q}_y \leqslant \mathbf{0}, \quad \forall \tau \in (0, \Delta t) \tag{24b}$$

$$\mathbf{S}\boldsymbol{\eta}_0 - \mathbf{f}_0 = \mathbf{0} \tag{24c}$$

$$\mathbf{S}\boldsymbol{\eta}_c - \hat{\mathbf{f}}_c = \mathbf{0}, \quad \forall \tau \in (0, \Delta t) \tag{24d}$$

$$\mathbf{C}^T\mathbf{Q}_\rho^s = \mathbf{0} \tag{24e}$$

$$\bar{\xi}_0 - \xi_0 \leqslant 0, \tag{24f}$$

where it has been set that

$$\mathbf{Q}^s = \xi_0\mathbf{Q}_0 + \xi\mathbf{Q}_c + \mathbf{Q}_\rho^s = \xi_0\mathbf{F}^{-1}\mathbf{C}\boldsymbol{\eta}_0 + \xi(\mathbf{F}^{-1}\mathbf{C}\boldsymbol{\eta}_c - \mathbf{F}^{-1}\mathbf{q}_\theta) + \mathbf{Q}_\rho^s. \tag{24g}$$

Following the Lagrange multiplier method, denoting by $e_c = (E_c^s/X) > 0$ a suitable constant, by $\dot{\boldsymbol{\lambda}}^s = \dot{\boldsymbol{\lambda}}^s(t) \geqslant \mathbf{0}$, $\xi_0\Delta\boldsymbol{\eta}_r^s$, $\xi\dot{\boldsymbol{\eta}}_r^s = \xi\dot{\boldsymbol{\eta}}_r^s(t)$, $\Delta\mathbf{u}^s$ and $e_0 = (E_0^s/\bar{\xi}_0) \geqslant 0$ the Lagrange multipliers (with factors ξ_0 and ξ as scaling factors not subjected to variations), the augmented functional of problem (24) is

$$\psi = -\xi e_c + \int_0^{\Delta t} (\xi_0\boldsymbol{\eta}_0^T\mathbf{C}^T\mathbf{F}^{-1}\mathbf{N} + \xi(\boldsymbol{\eta}_c^T\mathbf{C}^T\mathbf{F}^{-1} - \mathbf{q}_\theta^T\mathbf{F}^{-1})\mathbf{N} + (\mathbf{Q}_\rho^s)^T\mathbf{N} - \mathbf{Q}_y^T)\dot{\boldsymbol{\lambda}}^s \,\mathrm{d}\tau$$
$$- (\mathbf{S}\boldsymbol{\eta}_0 - \mathbf{f}_0)^T(\xi_0\Delta\boldsymbol{\eta}_r^s) - \int_0^{\Delta t} (\mathbf{S}\boldsymbol{\eta}_c - \hat{\mathbf{f}}_c)^T(\xi\dot{\boldsymbol{\eta}}_r^s)\,\mathrm{d}\tau - (\mathbf{Q}_\rho^s)^T\mathbf{C}\Delta\mathbf{u}^s + (\bar{\xi}_0 - \xi_0)e_0. \tag{25}$$

Taking the first variation of functional (25), with respect to all the variables, and since such functional must have a minimum with respect to the variables of problem (24) and a maximum with respect to the Lagrange multipliers, the relevant Kuhn–Tucker equations of problem (24) can be obtained; they are

$$\boldsymbol{\varphi}^s \leqslant \mathbf{0}, \quad \dot{\boldsymbol{\lambda}}^s \geqslant \mathbf{0}, \quad (\boldsymbol{\varphi}^s)^T\dot{\boldsymbol{\lambda}}^s = 0, \quad \forall \tau \in (0, \Delta t), \tag{26a}$$

$$\bar{\xi}_0 - \xi_0 \leqslant 0, \quad e_0 \geqslant 0, \quad (\bar{\xi}_0 - \xi_0)e_0 = 0 \tag{26b}$$

$$\mathbf{S}\boldsymbol{\eta}_0 - \mathbf{f}_0 = \mathbf{0} \tag{26c}$$

$$\mathbf{S}\boldsymbol{\eta}_c - \hat{\mathbf{f}}_c = \mathbf{0}, \quad \forall \tau \in (0, \Delta t) \tag{26d}$$

$$\mathbf{C}^T\mathbf{Q}_\rho^s = \mathbf{0} \tag{26e}$$

$$\mathbf{S}\dot{\boldsymbol{\eta}}_r^s - \mathbf{C}^T\mathbf{F}^{-1}\dot{\mathbf{q}}_p^s = \mathbf{0}, \quad \forall \tau \in (0, \Delta t) \tag{26f}$$

$$\mathbf{S}\Delta\boldsymbol{\eta}_r^s - \mathbf{C}^T\mathbf{F}^{-1}\Delta\mathbf{q}_p^s = \mathbf{0} \tag{26g}$$

$$\Delta\mathbf{q}_p^s - \mathbf{C}\Delta\mathbf{u}^s = \mathbf{0} \tag{26h}$$

$$\xi_0 \mathbf{f}_0^T \Delta \boldsymbol{\eta}_r^s = E_0^s \geqslant 0 \tag{26i}$$

$$X \int_0^{\Delta t} \hat{\mathbf{f}}_c^T \dot{\boldsymbol{\eta}}_r^s \, d\tau = E_c^s > 0, \tag{26j}$$

where it has been set

$$\dot{\mathbf{q}}_p^s = \mathbf{N}\dot{\boldsymbol{\lambda}}^s, \quad \forall \tau \in (0, \Delta t) \tag{27}$$

$$\Delta \mathbf{q}_p^s = \int_0^{\Delta t} \dot{\mathbf{q}}_p^s \, d\tau. \tag{28}$$

Equations (26a–j), (27) and (28) describe some kind of plastic deformation process, associated with the cyclic loading $\hat{\mathbf{f}}_x = \xi_0 \mathbf{f}_0 + X\hat{\mathbf{f}}_c$, in which $\dot{\mathbf{q}}_p^s$ plays the role of plastic strain rate vector, $\dot{\boldsymbol{\lambda}}^s$ the plastic multiplier vector, while $\dot{\boldsymbol{\eta}}_r^s$ the residual displacement rate vector. Equations (26a, c, d), (27) and (28) are formally identical to eqns (14a–c, e) and (15). Equation (26f) states that $\dot{\boldsymbol{\eta}}_r^s$ is the residual displacement rate vector generated by the plastic deformations $\dot{\mathbf{q}}_p^s$, and that both vary cyclically with period Δt. This occurrence implies that $\dot{\boldsymbol{\eta}}_r^s$ and $\dot{\mathbf{q}}_p^s$ are not compatible, that in the relevant truss a self-equilibrated residual stress rate vector $\dot{\mathbf{Q}}_r^s$, caused by $\dot{\mathbf{q}}_p^s$, is generated, to which a residual strain rate vector $\dot{\mathbf{q}}_r^s$, compatible with displacements $\dot{\boldsymbol{\eta}}_r^s$ (and then cyclically variable with period Δt), is associated. Equation (26f) is, therefore, equivalent to the following system

$$\dot{\mathbf{q}}_r^s = \mathbf{C}\dot{\boldsymbol{\eta}}_r^s, \quad \forall \tau \in (0, \Delta t) \quad \text{(compatibility)} \tag{29a}$$

$$\dot{\mathbf{Q}}_r^s = \mathbf{F}^{-1}(\dot{\mathbf{q}}_r^s - \dot{\mathbf{q}}_p^s), \quad \forall \tau \in (0, \Delta t) \quad \text{(elasticity law)} \tag{29b}$$

$$\mathbf{C}^T \dot{\mathbf{Q}}_r^s = \mathbf{0}, \quad \forall \tau \in (0, \Delta t) \quad \text{(equilibrium)}, \tag{29c}$$

where eqns (29b, a) are formally identical to eqns (14f, g), respectively. Since $\dot{\mathbf{q}}_r^s$ and $\dot{\mathbf{q}}_p^s$ vary cyclically, then the residual stress vector $\dot{\mathbf{Q}}_r^s$ must also vary cyclically, i.e. it must be

$$\int_0^{\Delta t} \dot{\mathbf{Q}}_r^s \, d\tau = 0 \tag{30}$$

as eqn (14h) explicitly states. By means of eqn (26g) the Lagrangian variable vector $\Delta \boldsymbol{\eta}_r^s$ takes the meaning of the increment in the cycle of residual displacements $\dot{\boldsymbol{\eta}}_r^s$, then one is able to write

$$\Delta \boldsymbol{\eta}_r^s = \int_0^{\Delta t} \dot{\boldsymbol{\eta}}_r^s \, d\tau. \tag{31}$$

Conversely, if definition (31) is true, by virtue of eqn (26f), we can also verify (26g). Equation (26h) states that the ratchet strain $\Delta \mathbf{q}_p^s$ is compatible with the displacement increments in the cycle $\Delta \mathbf{u}^s$; furthermore, we can take $\Delta \mathbf{u}^s = \Delta \boldsymbol{\eta}_r^s$ without harm, so that eqn (26h) gives results identical to eqn (17). Finally, setting

$$E^s = E_0^s + E_c^s = D^s, \tag{32}$$

where $D^s > 0$ is an arbitrary dimensional constant, by virtue of eqns (26i, j), one recognizes that eqn (32) is formally identical to eqns (18) and (22). It is very important to remark that constant D^+ introduced in Section 4 represents the gradient, with respect to the cyclic load multiplier, of the plastic dissipation in the cycle at the shakedown limit and has a very exact value, while constant D^s is a simple proportionality factor with an arbitrary value.

Within constants D^s and D^+, problems (26), (27), (28), (32), and (14a–h), (15)–(18), (22) are therefore equivalent, and their solutions are equal, within kinematical variables which differ by the ratio D^s/D^+ ($= E^s/E^+$). If, in particular, $E^s = D^s = 1$ one has

$$\dot{\boldsymbol{\lambda}}^+ = E^+\dot{\boldsymbol{\lambda}}^s, \quad \dot{\mathbf{q}}_p^+ = E^+\dot{\mathbf{q}}_p^s, \quad \forall\,\tau \in (0, \Delta t) \tag{33a}$$

$$\Delta\mathbf{q}_p^+ = E^+\Delta\mathbf{q}_p^s, \quad \Delta\boldsymbol{\eta}_r^+ = E^+\Delta\boldsymbol{\eta}_r^s. \tag{33b}$$

If the solution to problem (26a–j), (27), (28) provides $\xi_0 > \bar{\xi}_0$ (until now $\xi_0 = \bar{\xi}_0$ has been assumed), the shakedown limit load multiplier X shows insensitivity to the steady load variation, and this occurrence takes place when the point (ξ_0, X), in the plane (ξ_0, ξ), arrives at the so-called upper plateau of the line delimiting the shakedown domain (Polizzotto, 1993). In such a case, as it has already been pointed out for E_0^+ in Section 4, the external work produced by the steady load in the cycle, E_0^s, vanishes, while the solutions to problems (26), (27), (28), (32), and (14), (15)–(18), (22) differ not only for constants D^s and D^+, but also for the residual generalized stresses (the difference of which makes up for the difference of generalized stresses produced by the different steady loads).

6. THE BOUNDING PRINCIPLE

The steady-state elastic–plastic analysis of the structure subjected to the loads $\hat{\mathbf{f}}_\xi = \bar{\xi}_0\mathbf{f}_0 + \bar{\xi}\hat{\mathbf{f}}_c$, with $\bar{\xi} > X$, can be performed either through the solution to eqns (6a–h) or by a step-by-step full analysis of a convenient number of cycles until the steady-state phase is reached. In any case, these approaches involve a considerable computational effort. However, for loads slightly above the shakedown limit it is possible to establish what collapse mode is to be expected for the structure, minimizing the computational effort.

Equations (33b) allow us to establish what impending collapse mode happens at the shakedown limit, on the grounds of the solution to the shakedown load factor problem. If, for instance, $\Delta\mathbf{q}_p^s = \mathbf{0}$, this means that there is plastic shakedown, etc.

If the constant E^+ is known, eqns (33), transformed into

$$\dot{\boldsymbol{\lambda}} \cong E^+\dot{\boldsymbol{\lambda}}^s(\bar{\xi} - X), \quad \dot{\mathbf{q}}_p \cong E^+\dot{\mathbf{q}}_p^s(\bar{\xi} - X), \quad \forall\,\tau \in (0, \Delta t) \tag{34a}$$

$$\Delta\dot{\mathbf{q}}_p \cong E^+\Delta\dot{\mathbf{q}}_p^s(\bar{\xi} - X), \quad \Delta\boldsymbol{\eta}_r \cong E^+\Delta\boldsymbol{\eta}_r^s(\bar{\xi} - X) \tag{34b}$$

would allow us to calculate the plastic part of the steady-state response of the structure to loads $\hat{\mathbf{f}}_\xi$ slightly above the shakedown limit, by utilizing the solution to the shakedown cyclic load factor problem. Unfortunately, in order to compute E^+, by (23) for instance, the plastic strain rate vector $\dot{\mathbf{q}}_p$ must be known, i.e. the solution to the analysis problem must be known. This makes further investigations unnecessary.

In order to sustain a computational effort reduced to that necessary in order to solve the analysis problem, a suitable bounding technique may be employed. An upper bound E^* on the constant E^+ can be calculated, i.e.

$$E^+ \leqslant E^*. \tag{35}$$

Then, by eqns (34b) and (35), one can write

$$|\Delta\mathbf{q}_p| \leqslant E^*|\Delta\mathbf{q}_p^s|(\bar{\xi} - X), \qquad |\Delta\boldsymbol{\eta}_r| \leqslant E^*|\Delta\boldsymbol{\eta}_r^s|(\bar{\xi} - X). \tag{36}$$

Similarly one could do this for the quantities present in eqn (34a). Relations (36) and similar constitute bounds on the absolute value of the plastic part of the elastic–plastic response to loads $\hat{\mathbf{f}}_\xi$.

The expression of E^* can be obtained in the following manner. Let TS denote the given elastic–plastic truss structure subjected to load $\hat{\mathbf{f}}_\xi$, and constituted by a material for which associated plasticity laws hold (Martin, 1975) (as already described in Section 3). In addition to the compatibility and equilibrium equations, for all instants within the interval $(0, \Delta t)$, the following relationships have to be satisfied:

$$\mathbf{q} = \mathbf{q}_e + \mathbf{q}_\theta + \mathbf{q}_p \tag{37a}$$

$$\mathbf{q}_e = \mathbf{F}\mathbf{Q} \tag{37b}$$

$$\boldsymbol{\varphi} = \mathbf{N}^T\mathbf{Q} - \mathbf{Q}_y \leqslant \mathbf{0} \tag{37c}$$

$$\dot{\boldsymbol{\lambda}} \geqslant \mathbf{0}, \quad \dot{\mathbf{q}}_p = \mathbf{N}\dot{\boldsymbol{\lambda}} \tag{37d}$$

$$\boldsymbol{\varphi}^T\dot{\boldsymbol{\lambda}} = 0. \tag{37e}$$

To $\mathbf{Q}$ and $\dot{\mathbf{q}}_p$, corresponding through (37c–e), Drucker's postulate (Martin, 1975) can be applied

$$(\mathbf{Q}' - \mathbf{Q})^T\dot{\mathbf{q}}_p \leqslant 0, \quad \forall \tau \in (0, \Delta t), \tag{38}$$

where $\mathbf{Q}'$ is any stress vector, provided that it is plastically admissible, i.e. so as to satisfy (37c). It is worth noting that (38) is also valid element by element.

Let TS^* denote a fictitious truss. All the quantities related to TS^* are marked by the asterisk. TS^* is the same as TS, except for plastic behaviour: the complementarity constraint (37e) can be violated and the yield function is suitably perturbed. By symbols, for all instants within the interval $(0, \Delta t)$, one has

$$\mathbf{q}^* = \mathbf{q}_e^* + \mathbf{q}_\theta^* + \mathbf{q}_p^* \tag{39a}$$

$$\mathbf{q}_e^* = \mathbf{F}\mathbf{Q}^* \tag{39b}$$

$$\boldsymbol{\varphi}^* = \mathbf{N}^T\mathbf{Q}_\alpha^* - \mathbf{Q}_y = \mathbf{N}^T[\mathbf{Q}^* + \alpha(\xi_0\mathbf{Q}_0 + X\mathbf{Q}_c)] - \mathbf{Q}_y \leqslant \mathbf{0} \tag{39c}$$

$$\dot{\boldsymbol{\lambda}}^* \geqslant \mathbf{0}, \quad \dot{\mathbf{q}}_p^* = \mathbf{N}\dot{\boldsymbol{\lambda}}^*, \tag{39d}$$

where $\mathbf{q}_\theta^* = \mathbf{q}_\theta$ and the scalar $\alpha > 0$ is the multiplier of perturbation, which is the elastic stress response to load $\hat{\mathbf{f}}_X = \xi_0\mathbf{f}_0 + X\hat{\mathbf{f}}_c$. Because of the loosened form of (39), with respect to (37), the stress history $\mathbf{Q}^*$ in the cycle is not unique (as the $\mathbf{Q}$ one is); let us suppose that we know this history. To $\mathbf{Q}_\alpha^*$ and $\dot{\mathbf{q}}_p^*$, corresponding through (39c, d), we can apply a generalized Druker's postulate (Martin, 1975)

$$(\mathbf{Q}' - \mathbf{Q}_\alpha^*)^T\dot{\mathbf{q}}_p^* + (\boldsymbol{\varphi}^*)^T\dot{\boldsymbol{\lambda}}^* \leqslant 0, \quad \forall \tau \in (0, \Delta t), \tag{40}$$

where $\mathbf{Q}'$ is any stress vector, provided that it satisfies (39c). It is worth noting that (40) is also valid element by element. A material like the defined one, for which (37c, d) but not (37e) holds, or for which (40) but not (38) holds, is called pseudoplastic.

Stresses $\mathbf{Q}_\alpha^*$ are admissible in TS^*, but through eqn (37c) they also are admissible in TS, so that (38) can be applied with $\mathbf{Q}' = \mathbf{Q}_\alpha^*$

$$(\mathbf{Q}^* - \mathbf{Q})^T\dot{\mathbf{q}}_p + \alpha(\xi_0\mathbf{Q}_0 + X\mathbf{Q}_c)^T\dot{\mathbf{q}}_p \leqslant 0, \quad \forall \tau \in (0, \Delta t). \tag{41}$$

Stresses $\mathbf{Q}$ are admissible in TS, but through eqn (39c) they also are admissible in TS^*, so that (40) can be applied with $\mathbf{Q}' = \mathbf{Q}$

$$-(\mathbf{Q}^* - \mathbf{Q})^T\dot{\mathbf{q}}_p^* - \alpha(\xi_0\mathbf{Q}_0 + X\mathbf{Q}_c)^T\dot{\mathbf{q}}_p^* + (\boldsymbol{\varphi}^*)^T\dot{\boldsymbol{\lambda}}^* \leqslant 0, \quad \forall \tau \in (0, \Delta t). \tag{42}$$

By summing up (41) and (42) one obtains

$$(\xi_0\mathbf{Q}_0+X\mathbf{Q}_c)^T\dot{\mathbf{q}}_p \leqslant (\xi_0\mathbf{Q}_0+X\mathbf{Q}_c)^T\dot{\mathbf{q}}_p^* - \frac{1}{\alpha}(\boldsymbol{\varphi}^*)^T\dot{\boldsymbol{\lambda}}^* + \frac{1}{\alpha}(\mathbf{Q}^*-\mathbf{Q})^T(\dot{\mathbf{q}}_p^*-\dot{\mathbf{q}}_p), \quad \forall\,\tau\in(0,\Delta t) \tag{43}$$

which, integrated within the interval $(0, \Delta t)$, becomes

$$\int_0^{\Delta t}(\xi_0\mathbf{Q}_0+X\mathbf{Q}_c)^T\dot{\mathbf{q}}_p\,\mathrm{d}\tau \leqslant \int_0^{\Delta t}(\xi_0\mathbf{Q}_0+X\mathbf{Q}_c)^T\dot{\mathbf{q}}_p^*\,\mathrm{d}\tau - \frac{1}{\alpha}\int_0^{\Delta t}(\boldsymbol{\varphi}^*)^T\dot{\boldsymbol{\lambda}}^*\,\mathrm{d}\tau + \frac{1}{\alpha}\int_0^{\Delta t}(\mathbf{Q}^*-\mathbf{Q})^T(\dot{\mathbf{q}}_p^*-\dot{\mathbf{q}}_p)\,\mathrm{d}\tau. \tag{44}$$

Taking into account that

$$(\dot{\mathbf{q}}_p^*-\dot{\mathbf{q}}_p) = (\dot{\mathbf{q}}^*-\dot{\mathbf{q}})-\mathbf{F}(\dot{\mathbf{Q}}^*-\dot{\mathbf{Q}}) \tag{45}$$

the third integral on the right-hand side in (44) can be written as

$$\int_0^{\Delta t}(\mathbf{Q}^*-\mathbf{Q})^T(\dot{\mathbf{q}}_p^*-\dot{\mathbf{q}}_p)\,\mathrm{d}\tau = \int_0^{\Delta t}(\mathbf{Q}^*-\mathbf{Q})^T(\dot{\mathbf{q}}^*-\dot{\mathbf{q}})\,\mathrm{d}\tau - \int_0^{\Delta t}(\mathbf{Q}^*-\mathbf{Q})^T\mathbf{F}(\dot{\mathbf{Q}}^*-\dot{\mathbf{Q}})\,\mathrm{d}\tau = 0 \tag{46}$$

because by virtue of the principle of virtual work

$$(\mathbf{Q}^*-\mathbf{Q})^T(\dot{\mathbf{q}}^*-\dot{\mathbf{q}}) = 0, \quad \forall\,\tau\in(0,\Delta t) \tag{47}$$

and for the periodicity of $(\mathbf{Q}^*-\mathbf{Q})$

$$\int_0^{\Delta t}(\mathbf{Q}^*-\mathbf{Q})^T\mathbf{F}(\dot{\mathbf{Q}}^*-\dot{\mathbf{Q}})\,\mathrm{d}\tau = 0. \tag{48}$$

Equation (44), divided by $(\xi - X)$, taking into account (46), (22) and (23), takes the form

$$E^+ = E_0^+ + E_c^+ \cong \frac{\xi_0}{\xi-X}\mathbf{Q}_0^T\Delta\mathbf{q}_p + \frac{X}{\xi-X}\int_0^{\Delta t}\mathbf{Q}_c^T\dot{\mathbf{q}}_p\,\mathrm{d}\tau \leqslant \frac{\xi_0}{\xi-X}\mathbf{Q}_0^T\Delta\mathbf{q}_p^* + \frac{X}{\xi-X}\int_0^{\Delta t}\mathbf{Q}_c^T\dot{\mathbf{q}}_p^*\,\mathrm{d}\tau - \frac{1}{\alpha(\xi-X)}\int_0^{\Delta t}(\boldsymbol{\varphi}^*)^T\dot{\boldsymbol{\lambda}}^*\,\mathrm{d}\tau = E_0^*+E_c^*+E_\alpha^* = E^* \tag{49}$$

which is the searched bounding principle (35).

The computation of the right-hand side of (49) implies knowledge, for all $\tau\in(0,\Delta t)$, of the quantities $\dot{\boldsymbol{\lambda}}^*$, $\dot{\mathbf{q}}_p^*$, $\mathbf{Q}^*$ relative to the fictitious elastic–pseudoplastic process. These can be obtained by an elastic–pseudoplastic analysis, i.e. an incremental analysis with loose complementarity constraint (37e) (Polizzotto, 1989). Because the solution to such a problem is not unique, in practice one usually searches for a special solution, following optimality or quasi-optimality bound criteria (Giambanco *et al.* 1990a, 1992).

7. CONCLUSIONS

In the framework of elastic–perfectly plastic truss structures subjected to a combination of cyclic and steady loads, it has been shown that the solution to the shakedown load factor problem constitutes a limit solution (generalized stresses, generalized strains and displacements), which—within a proportionality factor—describes the gradient, with respect to the cyclic load multiplier, of the steady-state elastic–plastic truss response to loads at the shakedown limit. The proportionality factor is the external work of loads in a cycle and, therefore, it can be computed as a function of the generalized elastic stresses and generalized plastic strains, the knowledge of which implies having the solution to the steady-state analysis problem. However, the solution to the shakedown load factor problem can provide different information of practical interest in studying the steady-state elastic–plastic response to loads slightly above the shakedown limit. This alone allows us to establish the plastic collapse mode (plastic shakedown or ratcheting); together with the computation of the proportionality factor, it also allows us to check the precision of the analysis problem solution slightly above the shakedown limit; finally together with the computation of a bound on the proportionality factor, it allows us to compute bounds on the absolute value of any measure of real plastic deformation (sustaining a computational effort lower than the required one for a full incremental elastic–plastic analysis).

The proposed bounding technique can be extended to any other discrete or discretized structure. The choice of using compatible finite elements (until now implicitly utilized) may not be the best in order to solve the shakedown load factor problem (this is not the case for truss structures). In fact, self-stress punctual equilibrium may be violated. As a consequence, although the boundary principle remains valid, the bounds on plastic deformations, computed by means of the solution to the shakedown load factor problem, may be not very stringent. However, such bounds can be considered acceptable, especially in the first stage of structure design. A more stringent computation of the bounds can be obtained by means of a closer finite element discretization and/or by employing equilibrated or hybrid finite element models.

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